On a Korovkin-Type Theorem for Simultaneous Approximation

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1. INTRODUCTION

Several years ago, Knoop and Pottinger [9] (see also [20]) proved a quantitative Korovkin-type theorem for some positive linear operators acting on the Banach space $C^r(K)$ of real-valued and r-times continuously differentiable functions on a compact interval K of the real axis. Recently, the same authors proved [10] an analogous quantitative theorem for functions defined on an unbounded interval of the real axis. The purpose of the present paper is to generalize their first non-quantitative assertion for compact intervals [6, Korollar 2.2] by replacing the so-called almost-convexity property by the condition of convexity-preserving of two types of higher-order convex functions and by adding another condition in terms of Čebyšev norms and derivatives of order r. These two types of higher-order convex functions are some spline functions and some monosplines. Using a result of Bojanic and Roulier [2] (for a history of this result see the next section), one may give a system of r (types of) higher-order convex functions which may be used in Knoop–Pottinger's result but only when the operators L_n are applicabe also to functions in $C^{r-2}(K)$, say. However, if L_n, $n \geqslant 1$, act only on $C^{r-1}(K)$, just two more simpler types of higher-order convex functions will be indicated. These generalizations are motivated by the fact that even classical instances of positive linear operators such as those of Meyer-König and Zeller have the property of simultaneous approximation but do not entirely satisfy the conditions of Knoop's and Pottinger's result (see [13, 9]).

The present paper is organized in the following manner. The next section contains the notations and the basic concepts used in the sequel. In the third section we review the previous results on test function theorems on simultaneous approximation while in Section 4 the main results are stated and proved. In the last section we show that the main results are more general than the result of Knoop and Pottinger, among other remarks.

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2. PRELIMINARIES

Let $K = [a, b]$ be a compact interval of the real axis and let $K' \subset K$ be a subinterval of K. For brevity, we assume $a = 0$. We consider the Banach space $X = C^r(K)$ endowed with norm given by $\|g\|_X := \max_{0 \le j \le r}(\|D^j g\|_K)$. Here $\|\cdot\|_K$ denotes the Čebyšev norm in $C(K) := C^0(K)$ and D^j is the jth differential operator. The basic functions we will work with are the monomials of all degrees, some simple splines, and their linear combinations which represent some *monosplines*. Namely, for $i \in \mathbb{N} \cup \{0\}$, the function $e_i: K \ni x \mapsto x^i \in \mathbb{R}$ is the ith monomial and $\sigma_{i,c}: K \ni x \mapsto (x - c)^i_+ \in \mathbb{R}$, $c \in K$, is the so-called ith basic spline function. As usual,

$$(y)^i_+ = \begin{cases} 0 & \text{if } y < 0, \\ y^i & \text{if } y \ge 0. \end{cases}$$

As far as differentiability properties are concerned, e_i is smooth (i.e., $C^\infty(K) \ni e_i$) for any $i \ge 0$ and $\sigma_{i,c}$ is in $C^{i-1}(K)$ for any $i \ge 1$ but $\sigma_{i,c}$ is in $C^i(K)$ for no integer $i \ge 0$ and $c \in \text{Int } K$. Also we denote by $\prod_{r-1}$ the subspace of $C^r(K)$ spanned by $\{e_0, ..., e_{r-1}\}$.

Now we recall the notion of higher-order convexity. Let $\mathscr{K}^i_K := \{f \in C(K): [x_0, ..., x_i; f] \ge 0 \text{ for any } x_0 < \cdots < x_i \in K\}$, where the square brackets associated to the $(i+1)$-points from K and the function $f \in C(K)$ denote an ith order divided difference of f. The elements of $\mathscr{K}^i_K$ are called *convex functions of order* $(i-1)$ [19] or *i-convex functions* [2]. We note that when $f \in C^i(K)$, f is i-convex if and only if $D^i f \ge 0$ on K. We mention the following important particular cases: $i = 2$ represents the usual convex functions, the class $\mathscr{K}^1_K$ is the set of all nondecreasing functions on K, and the case $i = 0$ reduces to positive functions.

An operator $L: V \mapsto C(K')$, $V \subset C(K)$, is called *convex of order* $r-1$ ($r \ge 0$) if the following holds (cf. Lupas [13]):

$$f \in \mathscr{K}^r_K \cap V \text{ implies } Lf \in \mathscr{K}^r_{K'},$$

i.e., L is an r-convexity preserving operator.

Every n-convex function may be uniformly approximated by n-convex splines. This is a result (implicitly) contained in [18]. We note that Popoviciu used the term "fonctions elementaires" instead of "spline functions." See also Bojanic and Roulier [2], Tzimbalario [23], Dadu [4], and the recent papers by Kočić Lacković [6] and Cross [3]. In what follows we will refer especially to Bojanic and Roulier [2] where this property is applied to obtain necessary and sufficient conditions for a continuous linear operator to transform every continuous, n-convex function into an r-convex function. Namely [2, Theorem 2], a continuous

linear operator $L: C(K) \mapsto C(K')$ is convex of order $r - 1$ if and only if $(r \geq 2)$

$$[x_0, ..., x_r; Lp] = 0 \tag{2.1}$$

for every $p \in \prod_{r-1}$ and every set of $(r + 1)$ points $x_0 < \cdots < x_r$ from K and

$$L(\sigma_{r-1,c}) \in \mathscr{K}^r_{K'} \tag{2.2}$$

for every $c \in \text{Int } K$. Of course, this also holds for $r = 1$ if we assume that $L(\sigma_{0,c})$ has a meaning because $\sigma_{0,c}$ is not in $C(K)$ for $c \in \text{Int } K$.

A generalization of the above convexity notion for operators was given by Knoop and Pottinger [9, Definition 1.4]: an operator $L: V \to C(K')$ is called *almost convex of order* $r - 1$ $(r \geq 1)$ if there exist $p \geq 0$ integers i_j, $1 \leq j \leq p$, satisfying $0 \leq i_1 < \cdots < i_p < r$ such that

$$f \in \bigcap (\mathscr{K}^{i_j}_K : j = 1, ..., p) \cap \mathscr{K}^r_K \cap V$$

implies

$$Lf \in \mathscr{K}^r_{K'}.$$

Now using Theorem 3 of Bojanic and Roulier [2], a *sufficient* condition for the almost convexity of order $r - 1$ $(r \geq 2)$ of the continuous linear operator $L: C(K) \to C(K')$, $K = [0, b]$, is

$$Le_i \in \mathscr{K}^r_K \tag{2.3}$$

for every $i = 0, ..., r - 1$ and

$$L\sigma_{r-1,c} \in \mathscr{K}^r_{K'} \tag{2.4}$$

for every $c \in \text{Int } K$. The same remark as above for $r = 1$ holds, too.

3. REVIEW OF PREVIOUS RESULTS

We note that we are interested only in non-quantitative test function theorems on simultaneous approximation. For information concerning quantitative Korovkin-type assertions, the reader is referred to Gonska [5] (for compact intervals) and to Knoop and Pottinger [10] (for unbounded domains) and the references cited therein.

The classical result of Korovkin is mentioned for completeness in the following theorem.

THEOREM A. *Let $L_n: C(K) \to C(K')$ be a sequence of positive linear operators which verify*

$$\lim_n \|e_i - L_n e_i\|_{K'} = 0 \tag{3.1}$$

for $i = 0, 1, 2$. Then

$$\lim_n \|f - L_n f\|_{K'} = 0 \tag{3.2}$$

holds for any $f \in C(K)$.

A test function theorem on simultaneous approximation was first obtained by Sendov and Popov [21, 22] (cf. also [11, 15, 16, 12]). They proved

THEOREM B. *Let $L_n: C^r(K) \to C^r(K)$ be a sequence of positive linear operators which are convex of orders $0, 1, ..., r$ and satisfy (3.1). Then for each compact proper subinterval $K' \subset K$ we have*

$$\lim_n \|D^j f - D^j L_n f\|_{K'} = 0 \tag{3.3}$$

for $j = 0, 1, ..., r$ and for all $f \in C^r(K)$.

We note that Theorem 2 of Bojanic and Roulier mentioned in the above section cannot be used in Theorem B since $\sigma_{r-1,c}$ is in $C^r(K)$ for no number c from Int K. However, if the continuous linear operator $L: C^r(K) \to C^r(K')$ is convex of order $r - 1$ then (2.1) still holds (see [2, Proof of Theorem 2]). Because the condition (2.1) means that for every polynomial p of degree no greater than $r - 1$, Lp should be identically zero if $r = 0$ or a polynomial of degree no greater than $r - 1$ if $r \geqslant 1$, every continuous linear operator which is convex of order $r - 1$ preserves the polynomiality of the degree at most $r - 1$. Assuming this condition, replacing the convexity of orders $0, 1, ..., r$ of the operators L_n in Theorem B by their almost convexity (only) of order $r - 1$, and adding the other three test function conditions, Knoop and Pottinger [9, Korollar 2.2] have proved the following result:

THEOREM C. *Let $L_n: C^r(K) \to C^r(K')$ be a sequence of positive linear operators such that $L_n(\prod_{r-1}) \subseteq \prod_{r-1}$ for all n and each L_n is almost convex of order $r - 1$. Then, assuming (3.1) and*

$$\lim_n \|D^r e_{r+i} - D^r L_n e_{r+i}\|_{K'} = 0 \tag{3.4}$$

for $i = 0, 1, 2$, the relation (3.3) holds for any $f \in C^r(K)$.

Again, we cannot use convexity conditions (2.3) and (2.4) since $\sigma_{r-1,c}$ is not even in $C^{r-1}(K)$ for $c \in \text{Int } K$. Thus it is the aim of this paper to replace the functions e_i and $\sigma_{r-1,c}$ from (2.3) and (2.4) by essentially only two types of functions from $C^r(K)$, namely $\sigma_{r+1,c}$ and the monospline $\sigma_{r+1,c} + (cr+c) e_r - e_{r+1}$. Note that if $\sigma_{i,c} \in C^r(K)$ for all $c \in \text{Int } K$, then $i \geq r+1$, so the basic spline $\sigma_{r+1,c}$ is "minimal" in this sense. Also, if we assume that L_n are applicable to functions in $C^{r-1}(K)$ instead of only $C^r(K)$, then we may use the simpler functions $\sigma_{r,c}$ and $e_r - \sigma_{r,c}$ (see Theorem 2 below). Again, $\sigma_{r,c}$ has the above minimality property with respect to $C^{r-1}(K)$. However, due to these simplifications, a supplementary condition must be added, namely $\|D^r L_n g\|_{K'} \leq M \cdot \|D^r g\|_K$ for all sufficiently large n and all $g \in C^r(K)$, where M is independent of n and g. It implies $L_n(\prod_{r-1}) \subseteq \prod_{r-1}$. As Remark (i) in the last section will show, the above condition is indeed necessary for the property (3.3) of simultaneous approximation for a sequence of linear operators L_n verifying $L_n(\prod_{r-1}) \subseteq \prod_{r-1}$. Also if L_n verifies the conditions of Theorem C then the above condition is true. Thus we obtain a result which is a generalization of Theorem C and also we prove a simpler result under the additional assumption that each L_n is defined on $C^{r-1}(K)$.

4. Main Results

The following result is the announced generalization of Theorem C.

THEOREM 1. *Let* $L_n : C^r(K) \to C^r(K')$ *be a sequence of positive linear operators such that* $\|D^r L_n g\|_{K'} \leq M \cdot \|D^r g\|_K$ *for all* $g \in C^r(K)$, *where* M *does not depend on* n *and* g. *In addition, we assume that*

$$L_n \sigma_{r+1,c} \in \mathcal{K}^r_{K'} \tag{4.1}$$

$$L_n(\sigma_{r+1,c} + (cr+c) e_r - e_{r+1}) \in \mathcal{K}^r_{K'} \tag{4.2}$$

for all n *and all* $c \in \text{Int } K$. *Then if* (3.1) *and* (3.4) *are satisfied, the relation* (3.3) *holds for any* $f \in C^r(K)$.

Proof. First, we recall the convention $a = 0$. We define

$$I_r : C(K) \ni f \to \int_0^x ((x-t)^{r-1} f(t)/(r-1)!) \, dt \in C^r(K)$$

and then the linear operator $Q_n : C(K) \to C(K')$ by

$$Q_n := D^r \circ L_n \circ I_r. \tag{4.3}$$

Since L_n preserves polynomiality of degree at most $r-1$ it follows that

$L_n(f - I_r D'f) \in \prod_{r-1}$. Hence $D'L_n f = D'L_n I_r D'f$ and, keeping in mind (4.3), we arrive at

$$Q_n D'f = D'L_n f. \qquad (4.4)$$

Using (4.4) and the condition $\|D'L_n g\|_{K'} \leqslant M \cdot \|D'g\|_K$ we get

$$\|Q_n\| \leqslant M. \qquad (4.5)$$

Since $e_i = (i!/(r+i)!) D'e_{r+i}$, (4.4) and (3.4) imply that

$$\lim_n \|e_i - Q_n e_i\|_{K'} = 0 \qquad (4.6)$$

for $i = 0, 1, 2$. The above equality (4.6) shows that $Q_n e_i$ are bounded in $C(K')$ for $i = 0, 1, 2$ and $Q_n(e_1; x)/Q_n(e_0; x) \in K$ for $x \in K$ and all sufficiently large n. Hence, the following "perturbation" of Q_n defined on $C(K)$ by

$$Q'_n(f; x) := Q_n(f; x) - Q_n(e_0; x) f(Q_n(e_1; x)/Q_n(e_0; x))$$

is a well-defined linear operator for all sufficiently large n. In the remainder of this proof, n will be large enough to make all assertions below valid. Inequality (4.5) implies that

$$\|Q'_n\| \leqslant M', \qquad (4.7)$$

where M' is independent of n.

On the other hand, the convexity conditions (4.1) and (4.2) permit us to write, after some routine calculations, the inequality

$$D'L_n \sigma_{r+1,c} \geqslant (D'L_n(e_{r+1} - (cr+c) e_r))_+. \qquad (4.8)$$

But $D'\sigma_{r+1,c} = (r+1)! \sigma_{1,c}$ and $D'(e_{r+1} - (cr+c) e_r) = (r+1)! (e_1 - c)$ for all $r \geqslant 0$, $c \in \text{Int } K$, so (4.8) and (4.4) yield $Q_n \sigma_{1,c} \geqslant (Q_n(e_1 - c))_+$ which, in terms of the perturbed operators Q'_n, becomes

$$Q'_n \sigma_{1,c} \geqslant 0 \qquad (4.9)$$

for all $c \in \text{Int } K$. In (4.9) we have used also the inequality $Q_n e_0 > 0$ which follows from (4.6). The continuity of Q'_n, the relations $Q'_n e_i = 0$ for $i = 0, 1$, the inequality (4.9), and a result of Bojanic and Roulier [2, Theorem 1] show that

$$Q'_n f \geqslant 0 \qquad (4.10)$$

for every convex function $f \in C(K)$ and all (large) n.

Now, let f be a fixed element of $C^2(K)$. Then there are two real numbers $m^- = m^-(f)$ and $m^+ = m^+(f)$ such that the functions $f_1 := f - m^- e_2$ and $f_2 := m^+ e_2 - f$ are convex on K. Indeed, their second derivatives on K are given by $D^2 f_1 = D^2 f - 2m^-$ and $D^2 f_2 = 2m^+ - D^2 f$ and these are positive for certain suitable reals m^- and m^+ because $D^2 f$ is continuous and thus bounded on the compact interval K. Hence we may apply (4.10) for these particular convex functions in order to obtain $m^- Q'_n(e_2) \leqslant Q'_n(f) \leqslant m^+ Q'_n(e_2)$ or, keeping in mind (4.10) for $f = e_2$, the inequality

$$|Q'_n f| \leqslant |\mu| \cdot Q'_n e_2 \qquad (4.11)$$

with $\mu := \max(-m^-, m^+)$. But

$$|Q_n(f; x) - f(x)| \leqslant |Q_n(e_0; x) f(x) - f(x)| + |Q_n(f; x) - Q_n(e_0; x) f(x)|$$

$$\leqslant |f(x)| \cdot |Q_n(e_0; x) - e_0(x)| + |Q'_n(f; x)|$$

$$+ |Q_n(e_0; x)| \cdot |f(x) - f(Q_n(e_1; x)/Q_n(e_0; x))|,$$

so this and (4.11) yield the inequality

$$|Q_n(f; x) - f(x)| \leqslant |f(x)| \cdot |Q_n(e_0; x) - e_0(x)| + |\mu| \cdot |Q'_n(e_2; x)|$$

$$+ |Q_n(e_0; x)| \cdot |f(x) - f(Q_n(e_1; x)/Q_n(e_0; x))|. \qquad (4.12)$$

Because $Q'_n(e_2; x) = Q_n(e_2; x) - Q_n^2(e_1; x)/Q_n(e_0; x)$ and (4.6) holds for $i = 0, 1, 2$, the first two members of the right-hand part of (4.12) tend uniformly to zero when n tends to infinity. On the other hand f is continuous on K and thus uniformly continuous of the same interval. Hence $|f(x) - f(Q_n(e_1; x)/Q_n(e_0; x))| \to 0$ uniformly on K as $n \to \infty$. Since $|Q_n e_0|$ is bounded by (4.6), we get $\lim_n \|f - Q_n f\|_K = 0$ for any $f \in C^2(K)$. But $C^2(K)$ is a dense subspace of $C(K)$ and using (4.5) and the Banach–Steinhaus theorem we find $\lim_n \|f - Q_n f\|_K = 0$ for any $f \in C(K)$. Using again (4.4) we arrive at

$$\lim_n \| D^r f - D^r L_n f \|_K = 0 \qquad (4.13)$$

for every $f \in C^r(K)$. Now (2.1) and Theorem A of Korovkin imply that

$$\lim_n \|f - L_n f\|_K = 0 \qquad (4.14)$$

for every $f \in C(K)$.

Now the conclusion of Theorem 1 follows from (4.13), (4.14) and from the known fact that, on the Banach space $C^r(K')$, the two norms

$$g \to \max(\|g\|_{K'}, \|D^r g\|_{K'})$$

and

$$g \to \max(\|D^j g\|_{K'}; 0 \leqslant j \leqslant r)$$

are equivalent.

When L_n are applicable also to functions in $C^{r-1}(K)$ we may prove the following result.

THEOREM 2.   *Let* $L_n : C^{r-1}(K) \to C^r(K')$ *be a sequence of positive linear operators such that* $\|D^r L_n g\|_{K'} \leqslant M \cdot \|D^r g\|_K$, *where* M *does not depend on* n *and* g, *for all* n *and all* $g \in C^r(K)$. *In addition, we assume that*

$$L_n \sigma_{r,c} \in \mathscr{K}_{K'}^r \tag{4.15}$$

$$L_n(e_r - \sigma_{r,c}) \in \mathscr{K}_{K'}^r \tag{4.16}$$

for all $c \in K$. *Then if* (3.1) *and* (3.4) *are satisfied, the equality* (3.3) *holds for every* $f \in C^r(K)$.

Proof.   Let $M(K)$ be the set of bounded real functions on K with a most one point, say, of discontinuity in this interval and let $M^r(K)$ be the space of real-valued and r-times differentiable functions with $D^r f \in M(K)$. Now we may define the operators $I_r^* : M(K) \to C^{r-1}(K)$ and $Q_n^* : M(K) \to C(K')$ with the same form as those similar denoted and defined in the proof of Theorem 1. Also, let I_r and Q_n be the restrictions of I_r^* and Q_n^* to $C(K)$, respectively. Then, similarly as in Theorem 1, we have

$$Q_n D^r f = D^r L_n f \tag{4.17}$$

for any $f \in M^r(K)$. Equations (4.5) and (4.6) are also true. Now conditions (4.15) and (4.16) yield

$$0 \leqslant Q_n(e_0 - \sigma_{0,c}) \leqslant Q_n e_0 \tag{4.18}$$

since (4.17) holds and $D^r e_r = r! \, e_0$, $D^r \sigma_{r,c} = r! \, \sigma_{0,c}$. But, if $x \in K$ is fixed, the continuous linear functional

$$\varepsilon_x \circ Q_n : C(K) \ni f \to Q_n(f; x) \in \mathbb{R}$$

may be represented by Riesz's theorem as a Stieltjes integral in the following way,

$$Q_n(f; x) = \int_K f(t) \, d\lambda(t), \tag{4.19}$$

where the function $\lambda\colon K \ni t \to Q_n^*(e_0 - \sigma_{0,t}; x) \in \mathbb{R}$ is a function of bounded variation on K (see, for instance, [7, Chap. 6, Section 6]). Condition (4.18) means that $0 \leqslant \lambda(t) \leqslant \lambda(b)$ for all $t \in K$. But $\lambda(a) = 0$, so we have $\lambda(b) > \lambda(a)$. Hence λ verifies the assumptions of the Jensen–Steffensen inequality for convex functions (see, for instance, Pečarić [17]) and thus

$$ g\left(\int_K t \, d\lambda(t) \middle/ \int_K d\lambda(t)\right) \leqslant \left(\int_K g(t) \, d\lambda(t) \middle/ \int_K d\lambda(t)\right) \qquad (4.20) $$

for every convex function $g \in C(K)$. Using (4.19) we get

$$ Q_n(g; x) \geqslant Q_n(e_0; x) \, g(Q_n(e_1; x)/Q_n(e_0; x)). \qquad (4.21) $$

Because x was arbitrarily chosen in K, we obtain (4.10).

Now the proof may be completed as the proof of Theorem 1.

Another result of this type (but which is not a generalization of Theorem C) is contained in the following proposition.

THEOREM 3. *The same conclusion of Theorem 2 holds if* (4.15) *and* (4.16) *are replaced by*: *"there exists* $c_0 \in K$ *such that*

$$ L_n(\sigma_{r,c} - e_r) \in \mathscr{K}_{K'}^r \qquad (4.22) $$

for all $c \leqslant c_0$, $c \in K$, *and*

$$ L_n(-\sigma_{r,c}) \in \mathscr{K}_{K'}^r \qquad (4.23) $$

for all $c > c_0$, $c \in K$."

Proof. The proof is similar to the proof of Theorem 2, with (4.10) reversed and Theorem 1 of Pečarić [17] used instead of the Jensen–Steffensen inequality.

5. REMARKS

This section collects miscellaneous remarks on the topics discussed above.

(i) If each L_n is almost convex of order $r-1$, $L_n(\prod_{r-1}) \subseteq \prod_{r-1}$ and $\lim_n \|e_0 - L_n e_0\|_{K'} = 0$ (as is the case in Theorem C) then $\|D^r L_n g\|_{K'} \leqslant M \|D^r g\|_K$ for any $g \in C^r(K)$, where M is independent of n and g. Indeed, the above conditions show that the operators Q_n defined in the proof of Theorem 1 are linear and positive (see [9, 5]). Then $\|Q_n\| = \|Q_n e_0\|_{K'}$. Because Eq. (4.6) holds, we have $\|Q_n\| = \|Q_n e_0\|_{K'} \leqslant M$, where M is

independent of n. Thus for any $f \in C(K)$ we have $\|Q_n f\|_K \leqslant M \|f\|_K$ and by putting $f = D^r g$, $g \in C^r(K)$ we arrive (using also (4.4)) at the desired inequality. It is also worthwhile to mention that if the sequence of continuous linear operators (L_n) verifies (3.3) and $L_n(\prod_{r-1}) \subseteq \prod_{r-1}$, then $\|D^r L_n g\|_K \leqslant M \|D^r g\|_K$ with a suitable M. Indeed, under the above assumptions, the operators Q_n defined in Section 4 are linear and continuous and verify (4.4). Then, using (3.3), Q_n are approximating in $C(K)$. Therefore, by a classical result [8, Chap. 5, Sect. 1], there exists $M > 0$ independent of n such that $\|Q_n\| \leqslant M$ which implies the desired inequality.

(ii)   Now we prove that $\sigma_{r+1,c}$ and $g_{r+1,c}$ are in $\bigcap (\mathcal{K}'_K : 1 \leqslant i \leqslant r)$ for every $c \in \operatorname{Int} K$, $r \geqslant 0$, where $g_{r+1,c} := \sigma_{r+1,c} + (cr + c)e_r - e_{r+1}$ is the second function which appears in Theorem 1. Indeed, we have $D^i \sigma_{r+1,c} = (r+1)_i \sigma_{r-i+1,c}$ which is positive for all $0 \leqslant i \leqslant r$. Here $(a)_b := a(a-1)\cdots(a-b+1)$. Also, for $g_{r+1,c}$ we have $D^i g_{r+1,c} = (r+1)_i \sigma_{r-i+1,c} + (r+1)_i \{c(r-i+1)e_{r-i} - e_{r-i+1}\}$. Its positivity will be proved below. To this end, write for $x \leqslant c$, $x \in K$, $D^i g_{r+1,c}(x) = (r+1)_i x^{i-1}(c - x + c(r-i))$ which is positive for $0 \leqslant i \leqslant r$ and for $x > c$, $x \in K$ write

$$D^i g_{r+1,c}(x) = (r+1)_i x^{r-i+1}\{(1+y)^{r-i+1} - (1 + (r-i+1)y)\},$$

where $y := -c/x$. Because $x > c$, we get $y > -1$ and now the positivity of $D^i g_{r+1,c}(x)$, $x > c$, follows from Bernoulli's inequality $(1 + y)^n \geqslant 1 + ny$, $n \in \mathbb{N}$, $y > -1$.

The above two remarks show that our Theorem 1 is more general than Theorem C of Knoop and Pottinger.

(iii)   The construction of the functions f_1 and f_2 from the proof of Theorem 1 has been used first by Lupas [14] and then by other authors (see [1] for exact references and more remarks). For the convexity of the functions f_1 and f_2, the appartenence of the function f to the subspace $C^2(K)$ is essential; i.e., it cannot be replaced by $f | \operatorname{Int} K \in C^2(\operatorname{Int} K)$, $f \in C(K)$, for instance, as the following example due to Dr. I. Rasa (Cluj) shows. Let $K_0 := [0, 2]$ and $f_0 \in C(K_0)$ be the function given by

$$f_0 : K_0 \ni x \to (x-1)(1 - (x-1)^2)^{1/2} + \arcsin(x-1) \in \mathbb{R}.$$

Then f_0 is not derivable at the endpoints of K_0 and there are no finite real numbers m and m' such that f_1 or f_2 is convex on K_0, as again an investigation of their second derivates on $\operatorname{Int} K_0$ shows. In fact, for the convexity of f_1 and f_2 it would be sufficient to require that the initial function f is twice differentiable on the whole K and $D^2 f$ is bounded on the same interval.

(iv) There are also other inequalities of Jensen-type which may be used instead of Jensen–Steffensen's inequality in order to obtain different convexity conditions as in Theorems 2 and 3. For instance, we may use a generalization of the Jensen–Steffensen inequality, namely the so-called Jensen–Brunk inequality (cf. [17]), to obtain a generalization of Theorem 2. The price we paid for this extention is that the conditions of convexity are now more complicated, so we omit here the complete statement.

If in Theorem 3 we put $c_0 = a = 0$, then (4.22) and (4.23) reduce to $L_n(-\sigma_{r,c}) \in \mathcal{K}^r_{K'}$ for every $c \in K$; i.e., we obtain only one convexity condition. However, this is not a generalization of Theorem C since $-\sigma_{r,c}$ is not in $\bigcup (\mathcal{K}^i_K : 0 \leqslant i \leqslant r)$.

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